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MODELING HIV EPIDEMIC UNDER CONTACT TRACING- A REMARK

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ABSTRACT

A nonlinear model on HIV Epidemic under contact tracing is studied, where we assume that the rate of recruitment of HIV positives is proportional to the population. We determine the critical for the stability of the epidemic free equilibrium and the endemic equilibrium.

KEYWORDS: Contact Tracing, Basic Reproduction Number, Epidemic Free, Endemic

1. INTRODUCTION

Contact tracing has been used as a method of controlling of contagious disease [1,2]. While there is still a debate about contact tracing for HIV infection [3,4] the resurgence of infectious tuberculosis and outbreaks of drug resistance tuberculosis secondary to HIV induced immune depression is forcing many public health department to re-examine this policy [5,6].

2. MATHEMATICAL FORMULATION

We shall consider the differential system that describes the model which is proposed by Arazozo et al [7]. The model [7]

$$\begin{aligned}\frac{dN}{dt}&=-\alpha NX-\mu N+\delta\\ \frac{dX}{dt}&=\alpha NX-(k+\mu+\beta)X+\nu\\ \frac{dY}{dt}&=kX-(\mu+\beta)Y\\ \frac{dZ}{dt}&=\beta X+\beta Y-\mu' Z+\rho\end{aligned}\tag{1}$$

Where N- the population of sexually active person of susceptible

X- the number of HIV positives that do not know they are infected

Y- the number of HIV positive that know they are infected

Z- number of AIDS cases

k- the rate at which the HIV positives are detected.

β - the rate at which the HIV positive develops to AIDS

μ- the death rate of sexually active population

μ' - the death rate due to AIDS

δ - the recruitment to the classes of susceptible

ν - the immigration of unknown HIV positive

ρ - the immigration of AIDS cases

In this paper we assume that the rate of recruitment of HIV positive is proportional to the population,

The model becomes

$$\begin{aligned}\frac{dN}{dt}&=-\alpha NX-\mu N+\delta N\\ \frac{dX}{dt}&=\alpha NX-\left(k+\mu+\beta\right)X+\nu N\\ \frac{dY}{dt}&=kX-\left(\mu+\beta\right)Y\\ \frac{dZ}{dt}&=\beta X+\beta Y-\mu' Z+\rho\end{aligned} \tag{2}$$

After transformation (2) becomes

$$\begin{aligned}\frac{dN}{dt}&=-\alpha NX-\mu N+\delta N\\ \frac{dX}{dt}&=\alpha NX-\left(k+\mu+\beta\right)X+\nu N\\ \frac{dY}{dt}&=kX-\left(\mu+\beta\right)Y\\ \frac{dM}{dt}&=\beta X+\beta Y-\mu' M\end{aligned} \tag{3}$$

Descartes' Rule of Sign

The number of positive zeros of polynomial with real coefficients is either equals to the number of variations in sign of the polynomial or less than this by an even number [4].

3. RESULTS

- Stability of the Epidemic Free Equilibrium

Theorem 1: If $R_o < 1$, then the zero solution of the epidemic free equilibrium of (3) is asymptotically stable.

Proof: The Jacobian matrix of the epidemic equilibrium is

$$A=\begin{pmatrix} \delta-\mu & 0 & 0 & 0 \\ 0 & -(k+\mu+\beta) & 0 & 0 \\ 0 & k & -(\mu+\beta) & 0 \\ 0 & \beta & \beta & -\mu' \end{pmatrix}$$

$$\textit{The eigenvalues are } \lambda_1=\delta-\mu \quad (4)$$

$$\lambda_2=-(k+\mu+\beta) \quad (5)$$

$$\lambda_3=-(\mu+\beta) \quad (6)$$

$$\lambda_4=-\mu' \quad (7)$$

The zero epidemic equilibrium of (3) is asymptotically stable if (4) is less than zero $\left(i.e\ \lambda_1<0,\ if\ \ \delta-\mu<0\ then\ \delta<\mu, \frac{\delta}{\mu}<1, R_0<1\right)$

Theorem 2: If $R_o>1$, then the zero solution of the epidemic free equilibrium of (3) is unstable.

Proof: The Jacobian matrix of the epidemic equilibrium is

$$A=\begin{pmatrix} \delta-\mu & 0 & 0 & 0 \\ 0 & -(k+\mu+\beta) & 0 & 0 \\ 0 & k & -(\mu+\beta) & 0 \\ 0 & \beta & \beta & -\mu' \end{pmatrix}$$

The eigenvalues are

$$(\delta-\mu-\lambda)(-(k+\mu+\beta)-\lambda)(-(\mu+\beta)-\lambda)(-\mu'-\lambda)=0$$

$$\text{Re}\textit{call that } R_0=\frac{\delta}{\mu}\ \ ,\ R_0\mu=\delta$$

$$-\mu(R_0-1)(k+\mu+\beta)+\lambda(k+2\mu+\beta-\delta)+\lambda^2$$

$$\textit{and } \lambda_3=-(\mu+\beta)\ \ ,\ \lambda_4=-\mu'$$

$$\lambda^2+\lambda(k+\beta+\mu(2-R_0))-\mu(R_0-1)(k+\mu+\beta)=0$$

$$\textit{It is sufficient to show that at least one eigen value is positive. Now}$$

$$\lambda^2+\lambda(k+\beta+\mu(2-R_0))-\mu(R_0-1)(k+\mu+\beta)=0$$

$$\textit{has one variation in sign when}$$

$$(k+\beta+\mu)>0, \mu>0 \textit{ and } R_0>0.$$

Hence by Descartes' rule of sign, (8) has a positive root. This complete the proof.

- Stability of epidemic equilibrium

Theorem 3: If $r_3>0, r_2>0$, $r_1>0$ and $r_0>0$ the endemic equilibrium is asymptotically stable

Proof: The Jacobian matrix of epidemic equilibrium (3) is

$$A=\begin{pmatrix} 0 & -\frac{(k+\mu+\beta)(\delta-\mu)}{(-\mu+\delta+v)} & 0 & 0 \\ -\mu+\delta+v & v\frac{(k+\mu+\beta)}{(-\mu+\delta+v)} & 0 & 0 \\ 0 & k & -(\mu+\beta) & 0 \\ 0 & \beta & \beta & -\mu' \end{pmatrix}$$

The eigenvalues of the epidemic equilibrium is obtained by solving

$$|A-I\lambda|=0 \quad (9)$$

$$\lambda^4+\lambda^3 r_3+\lambda^2 r_2+\lambda r_1+r_0=0 \quad (10)$$

Where

$$A=\begin{pmatrix} 0 & -\frac{(k+\mu+\beta)(\delta-\mu)}{(-\mu+\delta+v)} & 0 & 0 \\ -\mu+\delta+v & v\frac{(k+\mu+\beta)}{(-\mu+\delta+v)} & 0 & 0 \\ 0 & k & -(\mu+\beta) & 0 \\ 0 & \beta & \beta & -\mu' \end{pmatrix}$$

$$r_3=(\mu'+\mu+\beta)-v\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$r_2=(k+\mu+\beta)(\delta-\mu)+\mu'(\mu+\beta)-v(\mu'+\mu+\beta)\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$r_1=(k+\mu+\beta)(\delta-\mu)(\mu'+\mu+\beta)-v\mu'(\mu+\beta)\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$r_0=(k+\mu+\beta)(\delta-\mu)\mu'(\mu+\beta)$$

(10) has a zero variation in signs. Hence by Descartes' rule of signs λ's are all negative roots and two complex number. If they are all negative, and then the critical point is asymptotically stable. Suppose

$\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta, \lambda_3 = -a \text{ and } \lambda_4 = -b$

we claim that $\alpha < 0$. *The number of complex roots will be* 2 *or* 0.

If there are two negative roots then

$$(\lambda - \alpha - i\beta)(\lambda - \alpha + i\beta)(\lambda + a)(\lambda + b) = 0 \text{ and}$$

$$\lambda^4 + (a - 2\alpha + b)\lambda^3 + \left(\alpha^2 + \beta^2 + ab - 2\alpha(a+b)\right)\lambda^2 + \left((a+b)\left(\alpha^2 + \beta^2\right) - 2ab\right)\lambda + ab\left(\alpha^2 + \beta^2\right) = 0$$

$$r_3 = a - 2\alpha + b$$

$$r_2 = \alpha^2 + \beta^2 + ab - 2\alpha(a+b)$$

$$r_1 = (a+b)\left(\alpha^2 + \beta^2\right) - 2ab$$

$$r_0 = ab\left(\alpha^2 + \beta^2\right)$$

But $r_3>0, r_{2>}>0$, $r_1>0$ and $r_0>0$ (given)

$$\therefore a - 2\alpha + b > 0, \alpha^2 + \beta^2 + ab - 2\alpha(a+b) > 0, (a+b)\left(\alpha^2 + \beta^2\right) - 2ab > 0, ab\left(\alpha^2 + \beta^2\right) > 0$$

Since “a” and “b” can be small as “a” $\to 0$, “b” $\to 0$ $r_1>0 \Rightarrow \alpha<0$

Hence the equilibrium point is asymptotically stable.

This completes the proof.

Theorem 4: If $R_0>1$, $\mu'(\delta + \nu) + \delta(\mu + \beta) > \mu\left(\mu' + \beta + \mu^2\right) + \nu k$,

$$(k + \mu + \beta)\left(\mu^2 + \delta^2 + \nu\delta\right) + \mu'\left((\beta + \mu)(\delta + \nu)\right) >$$
$$\nu(\mu' + \mu + \beta)(k + \mu + \beta) + \mu\left(\mu'(\beta + \mu) + (k + \mu + \beta)(2\delta + \nu)\right) \text{ and}$$
$$(\mu' + \mu + \beta)\left(\mu^2 + \delta^2 + \nu\delta\right) > \mu'\nu(\mu + \beta) + \mu(\mu' + \mu + \beta)(2\delta + \nu)$$

Then the endemic equilibrium is asymptotically stable.

Proof: The jacobian matrix of epidemic equilibrium (3) is

$$A = \begin{pmatrix} 0 & -\dfrac{(k+\mu+\beta)(\delta-\mu)}{(-\mu+\delta+\nu)} & 0 & 0 \\ -\mu+\delta+\nu & \nu\dfrac{(k+\mu+\beta)}{(-\mu+\delta+\nu)} & 0 & 0 \\ 0 & k & -(\mu+\beta) & 0 \\ 0 & \beta & \beta & -\mu' \end{pmatrix}$$

We want to show that $r_3>0, r_{2>}>0$, $r_1>0$ and $r_0>0$

$$r_3 = (\mu' + \mu + \beta) - v\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$(\mu' + \mu + \beta) - v\frac{(k+\mu+\beta)}{(-\mu+\delta+v)} = \frac{(\mu'+\mu+\beta)(-\mu+\delta+v) - v(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$= \frac{-\mu'\mu - \mu^2 - \mu\beta + \mu'\delta + \beta\delta + \mu'v + \mu v + \beta v - v(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$\frac{-\mu'\mu - \mu^2 - \mu\beta + \mu'\delta + \mu\delta + \beta\delta + \mu'v - vk}{(-\mu+\delta+v)} > 0$$

$$\text{since } \mu'(\delta+v) + \delta(\mu+\beta) > \mu(\mu'+\mu+\beta) + vk$$

$$\textit{this implies that } r_3 > 0$$

$$(k+\mu+\beta)(\delta-\mu) + \mu'(\mu+\beta) - v(\mu'+\mu+\beta)\frac{(k+\mu+\beta)}{(-\mu+\delta+v)} > 0$$

$$\frac{\mu'(\mu+\beta)(-\mu+\delta+v) + (k+\mu+\beta)(\delta-\mu)(-\mu+\delta+v) - v(\mu'+\mu+\beta)(k+\mu+\beta)}{(-\mu+\delta+v)} > 0$$

Clearly $r_2>0$ if $(k+\mu+\beta)(\mu^2+\delta^2+v\delta) + \mu'(\mu+\beta)(\delta+v)$

$$-\mu(\mu'(\mu+\beta) + (k+\mu+\beta)(2\delta-v)) - v(\mu'+\mu+\beta)(k+\mu+\beta) > 0$$

$$(k+\mu+\beta)(\mu^2+\delta^2+v\delta) + \mu'(\mu+\beta)(\delta+v) >$$

$$\mu(\mu'(\mu+\beta) + (k+\mu+\beta)(2\delta-v)) + v(\mu'+\mu+\beta)(k+\mu+\beta)$$

$$\textit{This implies that } r_2 > 0$$

Here we want to show that $r_1>0$

$$r_1 = (k+\mu+\beta)(\delta-\mu)(\mu'+\mu+\beta) - v\mu'(\mu+\beta)\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$\textit{Note that } (k+\mu+\beta)(\delta-\mu)(\mu'+\mu+\beta) - v\mu'(\mu+\beta)\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$= \frac{(k+\mu+\beta)(\delta-\mu)(\mu'+\mu+\beta)(-\mu+\delta+v) - v\mu'(\mu+\beta)(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$= \frac{(k+\mu+\beta)((\delta-\mu)(\mu'+\mu+\beta)(-\mu+\delta+v) - v\mu'(\mu+\beta))}{(-\mu+\delta+v)}$$

$$= (k+\mu+\beta)((\delta-\mu)(\mu'+\mu+\beta)(-\mu+\delta+v) - v\mu'(\mu+\beta)) > 0$$

$$(k+\mu+\beta) > 0 \textit{ and } (\delta-\mu)(\mu'+\mu+\beta)(-\mu+\delta+v) - v\mu'(\mu+\beta) > 0$$

$$(k+\mu+\beta) > 0 \textit{ and } -\mu(\mu'+\mu+\beta)(2\delta+v) - v\mu'(\mu+\beta) + (\mu'+\mu+\beta)(\mu^2+\delta^2+\delta v) > 0$$

$$\text{since } (\mu'+\mu+\beta)(\mu^2+\delta^2+\delta v) > \mu(\mu'+\mu+\beta)(2\delta+v) + v\mu'(\mu+\beta)$$

$$\textit{this implies that } r_1 > o$$

And lastly we want to show that $r_0>0$

$$r_0 = (k+\mu+\beta)(\delta-\mu)\mu'(\mu+\beta)$$

For r0>0,δ>μ

Implies that $R_o>1$

Hence $r_0>0$

4. DISCUSSIONS OF RESULTS

From the result obtained in (4) and (8), the basic reproduction number Ro is less than 1, then the epidemic free equilibrium is globally asymptotically stable but otherwise the epidemic is unstable i.e if $R_o>1$. From the result obtained in (10), the epidemic equilibrium is asymptotically stable if $R_o>1$. From what we had in (4) and (19) both the epidemic free and endemic equilibrium are stable and their stability depend on the basic reproduction number R_o. Therefore contact tracing could be used as a method of controlling the spread of the virus.

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